Maxwell's Equations

The Inconsistency in our Equations

Let's write the full set of equations we have come to:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o} \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{7.100}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_o \vec{J} \tag{7.101}$$

Now, we know that the divergence of a curl vanishes: it's a vector identity. We should check that it holds! For the electric field, we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = \vec{\nabla} \cdot -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{B} = 0 \tag{7.102}$$

If we repeat with $\vec{B}$, we obtain

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_o \vec{\nabla} \cdot \vec{J} = -\mu_o \frac{\partial \rho}{\partial t} \neq 0 \quad \text{in general} \tag{7.103}$$

There is a more physical way to see this by applying Ampere's Law to a circuit containing a parallel-plate capacitor. Construct an Ampere's Law loop around the wire carrying the current. Ampere's Law is satisfied because there is magnetic field in an azimuthal direction around the wire (giving a nonzero line integral of $\vec{B}$) and there is current passing through the disk-like surface whose boundary is the contour.

Now pick another surface that passes between the capacitor plates. This is an equally valid surface; nothing about our proof of Ampere's Law from the Biot-Savart Law assumed a particular choice of surface for the Ampere's Law surface integral. But this surface has no current intersecting it because it passes through the capacitor!

The reason this problem happens and we never noticed it before is because we have a case of non-steady currents here: charge piles up on the capacitor plates giving $\partial \rho / \partial t \neq 0$; we had assumed all along during magnetostatics and during our discussion of induction that $\partial \rho / \partial t = 0$.

The Displacement Current and Maxwell's Equations

In order to solve the above problem, we need something that will cancel

$$\mu_o\,\vec{\nabla}\cdot\vec{J} = -\mu_o\,\frac{\partial\rho}{\partial t} = -\mu_o\,\frac{\partial}{\partial t}\left(\epsilon_o\,\vec{\nabla}\cdot\vec{E}\right) = -\mu_o\,\vec{\nabla}\cdot\left(\epsilon_o\,\frac{\partial\vec{E}}{\partial t}\right) \tag{7.104}$$

Let's just add the necessary term to Ampere's Law:

$$\vec{\nabla}\times\vec{B} = \mu_o\,\vec{J} + \mu_o\,\epsilon_o\,\frac{\partial\vec{E}}{\partial t} \tag{7.105}$$

Was it ok to do this? Does it violate any of our previous conclusions? The only equation we have modified is the $\vec{\nabla}\times\vec{B}$ equation, so we only need to consider our study of magnetostatics, where we applied this equation. The addition preserves the usual behavior of $\vec{\nabla}\times\vec{B}$ for magnetostatics because $\partial\vec{E}/\partial t = 0$ in magnetostatics. Why? Two things can result in time dependence of $\vec{E}$. The first is time dependence in ρ. But in magnetostatics, we assume steady-state currents, explicitly requiring no buildup of charge and hence $\partial\rho/\partial t = 0$. The second is time dependence of $\vec{B}$, which can yield time dependence of $\vec{E}$ via Faraday's Law. But magnetostatics assumes $\vec{B}$ is constant in time, so there is no worry there.

The added term is called the *displacement current density*,

$$\boxed{\vec{J}_d \equiv \epsilon_o\,\frac{\partial\vec{E}}{\partial t}} \tag{7.106}$$

It is only called this because it appears in the Ampere's Law equation with the same units and form as a current. It is not a physical current carried by charges.